Relaxation from an intrinsically unstable state to the metastable state in a colored-noise-driven system

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The relaxation from an intrinsically unstable state to the metastable state of a bistable system, which is driven by colored noise, is investigated in the weak-noise strength D and general correlation time τ. The influence of τ on the relaxation of the system is discussed.

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I. INTRODUCTION

Systems driven by colored noise have recently activated a great deal of interest [1–15]. A generic and widely studied system is a bistable system described by the Langevin-like equation

$$\dot{x}=ax-x^3+\xi(t)\ , \tag{1.1}$$

with $a>0$, $\langle\xi(t)\rangle=0$, and

$$\langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp(-|t-t'|/\tau)\ . \tag{1.2}$$

The noise $\xi(t)$ itself undergoes an Ornstein-Uhlenbeck process, i.e.,

$$\dot{\xi}(t)=-\gamma\xi(t)+\Gamma(t)\ , \tag{1.3}$$

where

$$\langle\Gamma(t)\rangle=0,\quad \langle\Gamma(t)\Gamma(t')\rangle=2D\gamma^2\delta(t-t')\ , \tag{1.4}$$

and

$$\gamma=1/\tau\ . \tag{1.5}$$

We denote

$$\xi(t)=y(t); \tag{1.6}$$

then (1.1) and (1.3) are changed into

$$\dot{x}=ax-x^3+y,\quad \dot{y}=-\gamma y+\Gamma(t)\ . \tag{1.7}$$

Thus, the Fokker-Planck equation (FPE) corresponding to (1.7) reads

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}[(ax-x^3+y)P]+\frac{\partial}{\partial y}(\gamma yP)+D\gamma^2\frac{\partial^2}{\partial y^2}P\ . \tag{1.8}$$

The central difficulty of solving Eq. (1.8) is that such a FPE does not possess an explicit analytical expression, even for the stationary solution.

A general way to deal with this problem is to project Eq. (1.8) into an effective one-dimensional FPE, which can be written as

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}[C(x)P(x,t)]+\frac{\partial^2}{\partial x^2}[D(x)P(x,t)]\ . \tag{1.9}$$

Several well-known approximations of such type have been suggested; we here list some of them.

(i) Small-τ expansion [11]:

$$C(x)=f(x)=ax-x^3,\quad D(x)=D(1+\tau f')\ . \tag{1.10}$$

(ii) Unified theory [5,15]:

$$C(x)=[f+D\tau f''/(1-\tau f')^2]/(1-\tau f')\ ,\quad D(x)=D/(1-\tau f')^2\ . \tag{1.11}$$

(iii) Functional-calculus approximation [4]:

$$C(x)=f(x),\quad D(x)=D/(1-\tau f')\ . \tag{1.12}$$

(iv) Decoupling approximation [5]:

$$C(x)=f(x),\quad D(x)=D/(1-\tau\langle f'\rangle)\ . \tag{1.13}$$

In the last one, the average is to be taken over the unknown stationary distribution $P_{st}(x)$ self-consistently.

These effective approaches have been successfully used to produce the stationary probability distributions of the actual system in a wide range of correlation time τ [14,15]. However, the problem of whether these effective FPE's correctly describe the relaxation process has not been carefully studied. In spite of the fact that the mean first-passage time (MFPT) has been extensively discussed

[15], the problem of the evolution from an intrinsically unstable state to the metastable state in a colored-noise-driven system is rather seldom considered. In this paper, we are involved in the time-dependent problem of the FPE (1.8) in the weak-noise limit and for an arbitrary correlation time.

In dealing with the time-dependent problem, the Ω expansion theory (ΩET) [16,17], and the scaling theory (ST) [18,19] are often used. The ΩET is successful in describing the evolution of the system from an extensive region to a stable state, while failing in the region near an unstable point [16,19,20]. The ST is remarkable in characterizing the evolution from a one-peak distribution to a two-peak one, starting from an unstable point, but it has some trouble with the matching between various time regimes [18,20]. We here apply an approach called linear Ω expansion of a Green function (LΩEGF), which is suggested in [20–25], to discuss the relaxation of the colored-noise-driven system from an intrinsically unstable state to the metastable state. The result will be compared with those of the effective FPE's.

II. FROM AN UNSTABLE STATE TO THE METASTABLE STATE

A. General formulation of the LΩEGF

Because the ΩET is invalid near the unstable point, the central task is to get rid of this trouble. To do this, Hu and Zheng for the first time suggested a method named LΩEGF in Ref. [20]. The basic idea of this method is as follows.

(1) First, linearize the nonlinear drift term of the FPE around the unstable point, and solve the corresponding linearized FPE. In the weak-noise limit, this solution is a good approximation of the evolution of the real system in the initial time regime $t<t_s$; here t_s can be determined according to the specific model [see Refs. [20] and [24] and Eq. (2.8) in the present paper].

(2) In the time region around t_s, both the linearization approximation and the Ω expansion of the Green function are valid at the same time. Then, by using the linear approximation solution as the initial condition at time t_s, the Ω-expansion procedure to the Green function is performed; that produces the time evolution of the system up to the metastable state.

Following this procedure, we now consider the evolution of a colored-noise-driven system, which has been given in Sec. I, from an initially unstable state to the metastable state. Assuming that y is initially in its equilibrium distribution and x is located very near the unstable point (we locate x at $x=b\sqrt{D}$), so the initial distribution of the probability is prepared as

$$P(x,y,0)=\sqrt{1/2\pi D\gamma}\,\delta(x-b\sqrt{D}\,)\exp(-y^2/2D\gamma)\ , \tag{2.1}$$

with

$$b=O(1),\quad D\ll 1\ . \tag{2.2}$$

We first linearize the drift force in (1.8), leading to the following FPE:

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}[(ax+y)P]+\frac{\partial}{\partial y}(\gamma yP)+D\gamma^2\frac{\partial^2}{\partial y^2}P\ . \tag{2.3}$$

The time-dependent solution of (2.3) with the initial condition (2.1) reads [26]

$$P(x,y,t)=\frac{1}{2\pi}\left[\frac{1}{4\alpha_3\alpha_5-\alpha_4^2}\right]^{1/2}\times\exp\{[\alpha_5(x-\alpha_1)^2-\alpha_4(x-\alpha_1)(y-\alpha_2)+\alpha_3(y-\alpha_2)^2]/(4\alpha_3\alpha_5-\alpha_4^2)\}\ , \tag{2.4}$$

in which

$$\alpha_1=b\sqrt{D}\,e^{at}\ ,\quad \alpha_2=0\ ,$$

$$\alpha_3=-D\left[-\frac{1-e^{-2at}}{2a}+\frac{(1-e^{-(a+\gamma)t})}{\gamma+a}\right]\frac{\gamma e^{2at}}{\gamma-a}\ , \tag{2.5}$$

$$\alpha_4=\frac{D\gamma}{\gamma-a}(1-e^{(a-\gamma)t})\ ,\quad \alpha_5=\frac{D\gamma}{2}\ .$$

Later we will use this explicit solution as the initial condition when the Ω expansion procedure to the Green function is performed, and one will see some interesting conclusions can be drawn from it.

In the initial time regime

$$\exp(2at)\ll 1/D\ , \tag{2.6}$$

the solution (2.4) is a good approximation of the actual evolution of (1.8). In the case of

$$\exp(2at)\gg 1\ , \tag{2.7}$$

most of the probability flows out of the unstable region and then the Ω expansion of the Green function is desirable. Since $D\ll 1$, one may readily find a suitable time t_s satisfying

$$1\ll\exp(2at_s)\ll 1/D\ , \tag{2.8}$$

that the linearization of the drift and the Ω expansion of the Green function hold simultaneously [20–25]. The Green function of the system derived from Van Kampen's Ω expansion is

$$G(x,y,t)=\exp\{-[T-T(t)]\sigma^{-1}[T'-T'(t)]/2D\gamma^2\}\times[2\pi D\gamma^2|\det(\sigma)|^{1/2}]^{-1}\ , \tag{2.9}$$

where T and $T(t)$ are two-dimensional vectors

$$T=(x,y),\quad T(t)=[x(t),y(t)]\ , \tag{2.10}$$

and $T',T'(t)$ are their transpositions, respectively. σ is a matrix:

$$\sigma=\begin{bmatrix}\sigma_{xx}(t) & \sigma_{xy}(t)\\ \sigma_{xy}(t) & \sigma_{yy}(t)\end{bmatrix}\ , \tag{2.11}$$

of which the elements satisfy

$$\dot{\sigma}_{xx}=2(a-3x^2)\sigma_{xx}+2\sigma_{xy} ,$$
$$\dot{\sigma}_{xy}=\sigma_{yy}+(a-3x^2-\gamma)\sigma_{xy} , \tag{2.12}$$
$$\dot{\sigma}_{yy}=-2\gamma\sigma_{yy}+2 ,$$

and $x(t)$ and $y(t)$ are the solutions of

$$\dot{x}(t)=ax(t)-x(t)^3+y(t) ,$$
$$\dot{y}(t)=-\gamma y(t) , \tag{2.13}$$

with the initial conditions

$$x(t_s)=x_s , \quad y(t_s)=y_s ,$$
$$\sigma_{xx}(t_s)=\sigma_{xy}(t_s)=\sigma_{yy}(t_s)=0 . \tag{2.14}$$

Taking (2.4) as the initial distribution of (2.9) at t_s, we finally arrive at

$$P(x,y,t)=\int\int dx_s dy_s[4\pi^2(4\alpha_3\alpha_5-\alpha_4^2)]^{-1}\exp\{[\alpha_5(x_s-\alpha_1)^2-\alpha_4(x_s-\alpha_1)y_s+\alpha_3y_s^2]/(4\alpha_3\alpha_5-\alpha_4^2)\}$$
$$\times[2\pi D\gamma^2|\det(\sigma)|^{1/2}]^{-1}\exp\{-[T-T(t)]\sigma^{-1}[T'-T'(t)]/2D\gamma^2\} , \tag{2.15}$$

where α_i $(i=1,\ldots,5)$ are their values at time t_s.

As discussed above, the actual solution of (1.8) can be approximated by (2.4) at $t<t_s$ and (2.15) at $t>t_s$. By careful verification, we find that the integral (2.15) recovers (2.4) at $t<t_s$ (for the details of this matter, one can see Refs. [20] and [24]); thus, Eq. (2.15) provides the evolution of (1.8) from an intrinsically unstable state to the metastable state in the weak-noise limit. The problem of solving a two-dimensional FPE is reduced to the problem of solving the corresponding ordinary differential equations of (2.12) and (2.13), which are much easier than (1.8).

The efficiency of the LΩEGF method has been already verified for a one-dimensional bistable system subjected to a white noise in Ref. [20], where we use noise strengths $D=0.01$ and 0.05, which are not so small and are very practical for real physical-chemical systems. In Ref. [20], the coincidence between the exact numerical simulation and the analytical solution by LΩEGF is really striking. Nevertheless, this fact has not been verified for the two-dimensional case. By comparing the direct numerical solution of Eq. (1.8) with the integration of Eq. (2.15), one can test the efficiency of the approach. We expect that the LΩEGF will also give an accurate prediction of the actual $P(x,y,t)$.

B. Reduction of dimensions

Expressing (2.4) by the following form:

$$P(x',y',t)=[4\pi^2(4\alpha_3\alpha_5-\alpha_4^2)]^{-1}\times\exp(-x'^2/s-y'^2/w) , \tag{2.16}$$

is very convenient for understanding the feature of the probability distribution in the initial time region. In Eq. (2.16), we use the following transformation:

$$x'=[(x-\alpha_1)+k_1'y]/(1+k_1'^2)^{1/2} ,$$
$$y'=[(x-\alpha_1)+k_2'y]/(1+k_2'^2)^{1/2} , \tag{2.17}$$

where

$$k_{1,2}'=(\alpha_5-\alpha_3)/\alpha_4\pm[(\alpha_5-\alpha_3)^2/\alpha_4^2+1]^{1/2} , \tag{2.18}$$

and

$$s=-2\{\alpha_5+\alpha_3+\alpha_4[(\alpha_5-\alpha_3)^2/\alpha_4^2+1]^{1/2}\} ,$$
$$w=-2\{\alpha_5+\alpha_3-\alpha_4[(\alpha_5-\alpha_3)^2/\alpha_4^2+1]^{1/2}\} . \tag{2.19}$$

It is obvious that k_1' $(=1/k_2')$ varies with time, and the diffusion ellipse of the linear regime will rotate with time [25]. In the time region that satisfies $1\ll\exp(2at_s)\ll 1/D$, we have

$$k_2'\gg 1, \quad k_1'\ll 1 ,$$
$$s\approx-2\alpha_3\approx D\gamma e^{2at}/[a(\gamma+a)] , \tag{2.20}$$
$$w\approx-2\alpha_5\approx D\gamma ,$$

which indicate that

$$x'\approx x, \quad y'\approx y, \quad w\ll s . \tag{2.21}$$

This means that the diffusion ellipse develops, at time t_s, into a very narrow strip, which is much more elongated in the x direction than in the y direction, and thus the probability is practically distributed in one dimension. Therefore, we can treat (2.13) as a one-dimensional problem, i.e., the reduction of dimensions is practical [23–25].

Since the probability diffusion along the y axis can be neglected in comparison with that along the x axis, we can consider that the center of the Gaussian distribution of (2.9) is on the x axis, and then we reduce (2.13) into

$$\dot{y}(t)=0, \quad \dot{x}(t)=ax(t)-x(t)^3 , \tag{2.22}$$

and their solutions are

$$y(t)=0 ,$$
$$x(t)^2=-aC\exp(2at)/[1-C\exp(2at)] , \tag{2.23}$$

where

$$C=x_s^2\exp(-2at_s)/(x_s^2-a) . \tag{2.24}$$

By using (2.22) or (2.23), (2.12) can be solved with solutions

$$\sigma_{yy}(t)=\{1-\exp[2\gamma(t_s-t)]\}/\gamma \ ,$$
$$\sigma_{xy}(t)=\exp(-\gamma t)[ax(t)-x(t)^3] \times \int_{t_s}^{t} dt\, \sigma_{yy}(t)\exp(\gamma t)/[ax(t)-x(t)^3] \ , \quad (2.25)$$
$$\sigma_{xx}(t)=[ax(t)-x(t)^3]^2 \times \int_{t_s}^{t} 2[ax(t)-x(t)^3]^{-2}\sigma_{xy}(t)dt \ .$$

Thus, the two-dimensional FPE reduces to an integration with the integrant known explicitly.

The adiabatic approximation principle is widely used to reduce the dimensions of the problem. For the details of the application of this procedure to the colored-noise-driven system, one is referred to Refs. [12] and [13], where dimension-reduced FPE's were presented. Here the reduction of dimensions is applied for the relaxation from an intrinsically unstable state to the metastable state; the mechanism underlying the reduction is that, in the vicinity of an unstable point, the most unstable mode controls the system. (For the details, one is referred to Ref. [24].)

C. The probability distribution of the metastable state

As $t \to \infty$, from (2.22) and (2.12) we obtain

$$x(\infty)^2=a \ , \quad (2.26)$$

$$\sigma(\infty)^{-1}=\begin{bmatrix} 2a(2a+\gamma)^2 & 2a(2a+\gamma) \\ 2a(2a+\gamma) & 2a+\gamma \end{bmatrix} , \quad (2.27)$$

and

$$P(x,y,\infty)=\int dx_s(4\pi\alpha_3)^{-1/2}\exp[-(x_s-\alpha_1)^2/4\alpha_3]\{[2a\gamma(2a+\gamma)^2]^{1/2}/2\pi D\gamma^2\}$$
$$\times\{\exp[-a(2a+\gamma)^2(x+\sqrt{a})^2/D\gamma^2+2a(2a+\gamma)(x+\sqrt{a})y/D\gamma^2+(2a+\gamma)y^2/2D\gamma^2]$$
$$+\exp[-a(2a+\gamma)^2(x-\sqrt{a})^2/D\gamma^2+2a(2a+\gamma)(x-\sqrt{a})y/D\gamma^2+(2a+\gamma)y^2/2D\gamma^2]\} \ , \quad (2.28)$$

Thus, the final amount of probability acquired by the potential well of positive x reads

$$P^+=\int_{-b\sqrt{2a(1+a\tau)}/2}^{\infty}(1/\pi)^{1/2}\exp(-v^2)dv \ , \quad (2.29)$$

while the potential well of negative x acquires

$$P^-=\int_{-\infty}^{-b\sqrt{2a(1+a\tau)}/2}(1/\pi)^{1/2}\exp(-v^2)dv=1-P^+ \ . \quad (2.30)$$

If $\tau=0$, (2.29) and (2.30) return to the result of the white-noise case (see Ref. [24]). It is interesting to point out that the validity of (2.29) and (2.30) is guaranteed by the weak-noise condition $D \ll 1$, but it is applicable to the case of arbitrary τ. We can immediately find from (2.29) and (2.30) that the correlation time τ strongly affects the probability distribution of the metastable state. If $b>0$, i.e., the initial δ function is located in the positive side of the saddle point, the probability acquired by the potential well of $x=\sqrt{a}$ will increase as τ increases. When $\tau \to \infty$, almost all probability is obtained by this well, though the position of the initial δ function is not changed.

III. CONCLUSIONS AND DISCUSSIONS

In this paper, we use the LΩEGF to obtain a time-dependent solution of the FPE (1.8) with a bistable potential in the presence of colored noise in the weak-noise limit and arbitrary correlation time. By comparison, we discuss here the transient properties of the well-known effective FPE's listed in Sec. I.

Integrating (2.4) over y, we obtain

$$P(x,t)=\exp[-(x-\alpha_1)^2/4\alpha_3]/\sqrt{4\pi\alpha_3} \ , \quad (3.1)$$

in which α_1 and α_3 are given in (2.5). Given $f(x)=ax$ in Eqs. (1.10)–(1.13), the solutions of the effective FPE's can be written in the following form:

$$P(x,t)=\exp[-(x-\beta)^2/4\alpha]/\sqrt{4\pi\alpha} \ , \quad (3.2)$$

with

$$\text{(i)} \quad \alpha=-D(\gamma+a)(1-e^{2at})/2\gamma a, \quad \beta=x(0)e^{at} \ , \quad (3.3)$$

for the small-τ expansion;

$$\text{(ii)} \quad \alpha=-D\gamma\{1-\exp[2a\gamma t/(\gamma-a)]\}/2a(\gamma-a) \ , \quad \beta=x(0)\exp[a\gamma t/(\gamma-a)] \quad (3.4)$$

for the unified theory;

$$\text{(iii)} \quad \alpha=-D\gamma(1-e^{2at})/2a(\gamma-a), \quad \beta=x(0)e^{at} \quad (3.5)$$

for the functional-calculus approximation; and

$$\text{(iv)} \quad \alpha=D\gamma(1-e^{2at})/2a(\gamma-a), \quad \beta=x(0)e^{at} \quad (3.6)$$

for the decoupling approximation.

Thus, in the linear case, by comparison of (3.3), (3.4), (3.5), and (3.6) with (3.1), we conclude the following.

(a) For $a<0$, (3.3), (3.5), and (3.6) give correct characteristic relaxation times in the small-τ regime, while giving wrong ones in the large-τ regime. The unified theory produces incorrect results both in the characteristic time of variance α and the motion of the probability peak center β.

(b) For $a>0$, all the above effective FPE's produce wrong relaxation behaviors of the system, even in the small-τ regime.

Actually, in the linear case, an exact effective FPE has been derived by Fox's functional calculus [4], which is

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(axP)+\frac{D\gamma}{\gamma-a}[1-e^{(a-\gamma)t}]\frac{\partial^2}{\partial x^2}P \ , \quad (3.7)$$

where $P(x,t)$ also takes the same form as (3.2), with

$$\alpha=-\left[\frac{1-e^{-(a+\gamma)t}}{\gamma+a}-\frac{1-e^{-2at}}{2a}\right]\frac{D\gamma e^{2at}}{\gamma-a}=\alpha_3 \ ,$$
$$\beta=x(0)e^{2at} \ , \quad (3.8)$$

which is identical to our two-dimensional treatment.

As for the metastable state of the bistable model, we calculate, from (1.10), the probability gained by the two wells:

$$P^+=\int_{-b\sqrt{2a/(1+a\tau)}/2}^{\infty}\sqrt{1/\pi}\exp(-v^2)dv \ , \quad (3.9)$$

$$P^-=1-P^+ \ , \quad (3.10)$$

and from (1.11), (1.12), and (1.13), we obtain

$$P^+=\int_{-b\sqrt{2a(1-a\tau)}/2}^{\infty}\sqrt{1/\pi}\exp(-v^2)dv \ , \quad (3.11)$$

$$P^-=1-P^+ \ , \quad (3.12)$$

where we assume $x(0)=b\sqrt{D}$. Comparing (3.9), (3.10), (3.11), and (3.12) with (2.29) and (2.30), we can easily find that, only when $\tau=0$, they come to the same result, which is just that of the white-noise case [20]. Thus, all the effective FPE's described above produce incorrect results of the relaxation from an intrinsically unstable state to the metastable states, even for small τ, because of the wrong correction terms, though they produce good approximations of the stationary distribution.

In this presentation, we have considered only the evolution from the intrinsically unstable state to the metastable state. In the weak-noise limit $D \ll 1$, the metastable state has an extremely long lifetime and is a practical state observable. The probability evolution to the stationary state have been analyzed, for instance, in Refs. [14] and [15]. It is found that, for strongly colored noise, a probability "hole" may occur in the vicinity of the origin, which essentially influences the first passage time as well as the evolution from the metastable state to the stationary state. Nevertheless, this interesting fact is beyond the scope of our present paper.

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